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On the Integrability of Double Cosine and Sine Series, II

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We consider double cosine and sine series whose coefficients form a null sequence of bounded variation. In this case, a double cosine (or sine) series converges to a function $f(x, y)$ (or $g(x, y)$, respectively). The convergence can be understood even in the regular sense introduced by Hardy, and this implies convergence in Pringsheim's sense, too. We give sufficient conditions under which $f(x, y)/xy$ and $g(x, y)$ are integrable on $[0, \pi] \times [0, \pi]$ in the sense of improper Riemann integral. We conjecture that these conditions are essentially necessary. Our results are the extensions of those by R. P. Boas from one-dimensional to two-dimensional trigonometric series. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let $\{a_{jk}\}$ be a double sequence of real numbers such that

$$a_{jk} \to 0 \qquad \text{as} \quad j + k \to \infty \tag{1.1}$$

and

$$\sum_j \sum_k |\Delta_{11} a_{jk}| < \infty, \tag{1.2}$$

where j, k run over either 0, 1, ... or 1, 2, ... independently of one another. Here

$$\Delta_{11} a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.$$

We also use the notations

$$\Delta_{10} a_{jk} = a_{jk} - a_{j+1,k}, \qquad \Delta_{01} a_{jk} = a_{jk} - a_{j,k+1}.$$

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Under conditions (1.1) and (1.2), the double cosine series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \cos ky = f(x, y), \tag{1.3}$$

where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ for $j \geqslant 1$, converges for all $0 < x, y \leqslant \pi$; while the double sine series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky = g(x, y) \tag{1.4}$$

converges for all x, y. The pointwise convergence in (1.3) and (1.4) is meant in Pringsheim's sense. (See, e.g., [6, Vol. 2, Chap. 17].) Even more is true: the series in (1.3) and (1.4) converge regularly. Concerning the notion of regular convergence, we refer the reader to [2] and also [3]. As to the proof of regular convergence of series (1.3) and (1.4) under conditions (1.1) and (1.2), see [4] where proof is carried out in the case of the complex trigonometric system.

In the sequel, we denote by $Q = [0, \pi] \times [0, \pi]$ the positive quadrant of the two-dimensional torus.

2. Main Results

We have proved in [5] the following theorems on the integrability of series (1.3) and (1.4).

THEOREM A. *If conditions* (1.1) *and* (1.2) *are satisfied, then*

(i) *f is integrable on Q in the sense of improper Riemann integral, and*

(ii) *series* (1.3) *is the Fourier series of f in the same sense.*

Concerning the integrability of series (1.4), we restricted ourselves to the particular case

$$\Delta_{11} a_{jk} \geqslant 0 \qquad (j, k = 1, 2, ...). \tag{2.1}$$

THEOREM B. *If conditions* (1.1) *and* (2.1) *are satisfied, then g is integrable on Q in the sense of improper Riemann integral if and only if*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} < \infty. \tag{2.2}$$

We note that under conditions (1.1) and (1.2), $g(x)\sin jx \sin ky$ is continuous for all $j, k = 1, 2, \ldots$ and series (1.4) is the generalized Fourier sine series of g (according to the terminology in [6, Vol. 1, p. 48]).

Since (1.1) and (2.1) imply that

$$a_{jk} \geqslant 0, \qquad \Delta_{10} a_{jk} \geqslant 0, \qquad \Delta_{01} a_{jk} \geqslant 0 \quad (j, k = 1, 2, \ldots),$$

Theorem B has a limited scope of applications. In this paper, we treat the problem of the integrability of g in the general case. However, we have to assume a stronger condition than (1.2), namely

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\Delta_{11} a_{jk}| \ln(\max(j, k) + 1) < \infty. \tag{2.3}$$

Our main result reads as follows.

THEOREM 1. *If conditions* (1.1) *and* (2.3) *are satisfied and the series*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \qquad \textit{converges regularly}, \tag{2.4}$$

then the improper integral

$$\lim_{\delta, \varepsilon \downarrow 0} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} g(x, y)\, dx\, dy \qquad \textit{exists}. \tag{2.5}$$

Remark 1. We raise two problems in connection with Theorem 1.

(i) As a by-product of the proof of Theorem 1, we can conclude (cf. (4.2), (4.3), (4.11), and (4.16) below) that

$$\lim_{\delta, \varepsilon \downarrow 0} \left\{ \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} g(x, y)\, dx\, dy - \sum_{j=1}^{[1/\delta]} \sum_{k=1}^{[1/\varepsilon]} \frac{a_{jk}}{jk} (1 - \cos j\pi)(1 - \cos k\pi) \right\} = 0,$$

where $[u]$ denotes the greatest integral part of u. This makes it plausible that condition (2.4) is not only sufficient but also necessary for the fulfillment of (2.5). However, we are unable to present a rigorous proof of this reasonable conjecture.

We emphasize that the conditions in Theorem 1 imply more than (2.5). To go into details, we consider the so-called "row" series (i.e., when k is fixed and delete $\sin ky$ in series (1.4))

$$\sum_{j=1}^{\infty} a_{jk} \sin jx = g_k(x) \qquad (k = 1, 2, \ldots) \tag{2.6}$$

and "column" series

$$\sum_{k=1}^{\infty} a_{jk} \sin ky = h_j(y) \qquad (j = 1, 2, ...). \tag{2.7}$$

The pointwise convergence of series in (2.6) and (2.7) follows from the fact that the single sequences $\{a_{jk} : j = 1, 2, ...\}$ and $\{a_{jk} : k = 1, 2, ...\}$ are null sequences of bounded variation for each fixed k or j, respectively, due to (1.1) and (1.2). By virtue of the corresponding one-dimensional result of Boas [1],

$$\lim_{\delta \downarrow 0} \int_{\delta}^{\pi} g_k(x)\, dx \qquad \text{exists} \tag{2.8}$$

if and only if

$$\sum_{j=1}^{\infty} \frac{a_{jk}}{j} \qquad \text{converges} \quad (k = 1, 2, ...);$$

and similarly,

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\pi} h_j(y)\, dy \qquad \text{exists} \tag{2.9}$$

if and only if

$$\sum_{k=1}^{\infty} \frac{a_{jk}}{k} \qquad \text{converges} \quad (j = 1, 2, ...).$$

Since the regular convergence of series (2.4) implies the convergence of its rows and columns, we can conclude (2.8) and (2.9) (together with (2.5)) from the conditions in Theorem 1.

Thus, it is quite natural to assume the fulfillment of (2.8) and (2.9) together with that of (2.5) when we try to prove the converse implication in Theorem 1. Then, by the one-dimensional result of Boas [1] mentioned above it follows that each row and each column of series (2.4) must converge. It remains only to prove the convergence of the series in (2.4) in Pringsheim's sense. But at this point we are unable to proceed.

(ii) It is also an open problem whether the logarithmic factor in condition (2.3) is essential for the validity of Theorem 1 or not. In a weaker version of this problem, we may ask if there exists a double sequence $\{a_{jk}\}$ such that conditions (1.1), (1.2), and (2.4) are satisfied, but (2.5) fails. By

Theorem B, if the a_{jk} are of constant sign, then $\{a_{jk}\}$ cannot be taken as a counterexample.

The integrability of $g(x, y)/xy$ was also settled in [4] in case the series (1.4) which defines g converges absolutely.

THEOREM C. *If*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}| < \infty,$$

then the improper integral

$$\lim_{\delta, \varepsilon \downarrow 0} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \frac{g(x, y)}{xy}\, dx\, dy \qquad \textit{exists.}$$

In this paper, we will deal with the more intricate question of the integrability of $f(x, y)/xy$. Owing to (2.3), we have to assume that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}| \ln(\max(j, k) + 2) < \infty, \tag{2.10}$$

a stronger condition than the absolute convergence of series (1.3) which defines f.

It is easy to obtain natural preassumptions. If $f(x, y)/xy$ is integrable on Q in the sense of improper Riemann integral, then we have necessarily

$$f(x, 0) = 0 \qquad \text{and} \qquad f(0, y) = 0$$

for all x and y, respectively. Since

$$f(x, 0) = \sum_{j=0}^{\infty} \lambda_j \left(\sum_{k=0}^{\infty} \lambda_k a_{jk} \right) \cos jx$$

we conclude that

$$\sum_{k=0}^{\infty} \lambda_k a_{jk} = 0 \qquad (j = 0, 1, ...). \tag{2.11}$$

Analogously, it follows that

$$\sum_{j=0}^{\infty} \lambda_j a_{jk} = 0 \qquad (k = 0, 1, ...). \tag{2.12}$$

Obviously, (2.11) and (2.12) imply that $f(0, 0) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} = 0$.

THEOREM 2. *If conditions* (2.10)–(2.12) *are satisfied and*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \qquad \textit{converges regularly,} \tag{2.13}$$

then the improper integral

$$\lim_{\delta, \varepsilon \downarrow 0} \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \frac{f(x, y)}{xy}\, dx\, dy \qquad \textit{exists.} \tag{2.14}$$

Remark 2. If we assume the fulfillment of (2.10)–(2.12), then condition (2.13) and the condition that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(m+\alpha)(n+\beta)} \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \qquad \text{converges regularly}$$

are equivalent for any fixed $0 < \alpha,\ \beta < 1$.

Remark 3. We raise two more open problems in connection with Theorem 2.

(iii) We consider the row series

$$\sum_{j=0}^{\infty} \lambda_j a_{jk} \cos jx = f_k(x) \qquad (k = 0, 1, ...) \tag{2.15}$$

and column series

$$\sum_{k=0}^{\infty} \lambda_k a_{jk} \cos ky = e_j(y) \qquad (j = 0, 1, ...) \tag{2.16}$$

of the double series (1.3). By virtue of the corresponding one-dimensional result of Boas [1] (observe that conditions (2.11) and (2.12) are essential)

$$\lim_{\delta \downarrow 0} \int_{\delta}^{\pi} \frac{f_k(x)}{x}\, dx \qquad \text{exists} \tag{2.17}$$

if and only if

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{j=0}^{m} \lambda_j a_{jk} \qquad \text{converges} \quad (k = 0, 1, ...);$$

and similarly,

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\pi} \frac{e_j(y)}{y}\, dy \qquad \text{exists} \tag{2.18}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} \lambda_k a_{jk} \qquad \text{converges} \quad (j=0, 1, ...).$$

Since the regular convergence of series (2.13) implies the convergence of its rows and columns, the conditions of Theorem 2 actually imply the fulfillment of (2.17) and (2.18), too.

Now, if we want to prove the necessity of condition (2.13), it is appropriate to start with the assumption of conditions (2.14), (2.17), and (2.18). Again, we can immediately deduce the convergence of each row and column of series (2.13). But we are unable to infer the convergence of the series in (2.13) in Pringsheim's sense.

(iv) It is also an open problem whether the logarithmic factor in (2.10) is essential for the validity of Theorem 2 or not. (Cf. (ii) in Remark 1.)

On closing, we note that Theorems 1 and 2 can be considered as the extensions of the corresponding theorems of Boas [1] from one-dimensional to two-dimensional trigonometric series.

3. Auxiliary Results

We need five lemmas.

Let $\{b_j : j=0, 1, ...\}$ and $\{c_j\}$ be sequences of real numbers.

Lemma 1 (known). *If* $c_j > 0$ $(j=0, 1, ...)$ *and*

$$B_M = \sum_{j=m}^{M} b_j \qquad (0 \leqslant m \leqslant M),$$

then

$$\left| \sum_{j=m}^{M} b_j c_j \right| \leqslant \begin{cases} 2c_M \max |B_{m_1}| & \text{if } \{c_j\} \text{ is nondecreasing;} \\ c_m \max |B_{m_1}| & \text{if } \{c_j\} \text{ is nonincreasing;} \end{cases}$$

where the maximum is extended over $m \leqslant m_1 \leqslant M$.

Next, let $\{b_{jk} : j, k=0, 1, ...\}$ and $\{c_{jk}\}$ be double sequences of real numbers.

LEMMA 2. *If* $c_{jk} > 0$ $(j, k = 0, 1, ...)$ *and*

$$B_{MN} = \sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk} \qquad (0 \leqslant m \leqslant M, 0 \leqslant n \leqslant N),$$

then

$$\left| \sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk} c_{jk} \right|$$

$$\leqslant \begin{cases} 4c_{MN} \max |B_{m_1,n_1}| & \textit{if } \{c_{jk}\} \textit{ is nondecreasing in } j \textit{ and } k, \textit{ and } \Delta_{11} c_{jk} \geqslant 0; \\ 2c_{MN} \max |B_{m_1,n_1}| & \textit{if } \{c_{jk}\} \textit{ is nondecreasing and } \Delta_{11} c_{jk} \leqslant 0; \\ c_{mn} \max |B_{m_1,n_1}| & \textit{if } \{c_{jk}\} \textit{ is nonincreasing and } \Delta_{11} c_{jk} \geqslant 0; \end{cases}$$

where the maximum is extended over $m \leqslant m_1 \leqslant M$ *and* $n \leqslant n_1 \leqslant N$.

Proof. It is clear if we perform a double summation by parts:

$$\sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk} c_{jk} = \sum_{j=m}^{M-1} \sum_{k=n}^{N-1} B_{jk} \Delta_{11} c_{jk} + \sum_{j=m}^{M-1} B_{jN} \Delta_{10} c_{jN} + \sum_{k=n}^{N-1} B_{Mk} \Delta_{01} c_{Mk} + B_{MN} c_{MN}.$$

LEMMA 3. *If conditions* (1.1) *and* (1.2) *are satisfied, then there exist sequences* $\{b_{jk}\}$ *and* $\{c_{jk}\}$ *such that* $a_{jk} = b_{jk} - c_{jk}$,

$$b_{jk} \to 0 \quad \textit{and} \quad c_{jk} \to 0 \quad \textit{as} \quad j + k \to \infty,$$

$$\Delta_{11} b_{jk} \geqslant 0 \quad \textit{and} \quad \Delta_{11} c_{jk} \geqslant 0,$$

$$|b_{jk}|, |c_{jk}| \leqslant \sum_{p=j}^{\infty} \sum_{q=k}^{\infty} |\Delta_{11} a_{pq}|.$$

Remark 4. It follows that $b_{jk} \geqslant 0$, $c_{jk} \geqslant 0$,

$$\Delta_{10} b_{jk} \geqslant 0, \qquad \Delta_{01} b_{jk} \geqslant 0, \qquad \Delta_{10} c_{jk} \geqslant 0, \qquad \Delta_{01} c_{jk} \geqslant 0.$$

Proof. It is obvious by setting

$$b_{jk} = \frac{1}{2} \sum_{p=j}^{\infty} \sum_{q=k}^{\infty} (|\Delta_{11} a_{pq}| + \Delta_{11} a_{pq}),$$

$$c_{jk} = \frac{1}{2} \sum_{p=j}^{\infty} \sum_{q=k}^{\infty} (|\Delta_{11} a_{pq}| - \Delta_{11} a_{pq}).$$

LEMMA 4. *If conditions* (1.1) *and* (1.2) *are satisfied, then* $\{a_{jk}/jk\}$ *is also a null sequence of bounded variation. If conditions* (1.1) *and* (2.3) *are satisfied, then*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} j \left| \Delta_{11} \left(\frac{a_{jk}}{jk} \right) \right| < \infty. \tag{3.1}$$

Proof. It is easy to check that

$$\Delta_{11} \left(\frac{a_{jk}}{jk} \right) = \frac{\Delta_{11} a_{jk}}{jk} + \frac{\Delta_{10} a_{j,k+1}}{jk(k+1)} + \frac{\Delta_{01} a_{j+1,k}}{j(j+1)k} + \frac{a_{j+1,k+1}}{j(j+1)k(k+1)}$$

and the first statement in Lemma 4 is proved in a routine way.

To prove (3.1), we use the inequality

$$|\Delta_{10} a_{j,k+1}| \leqslant \sum_{q=k+1}^{\infty} |\Delta_{11} a_{jq}|$$

whence, by (1.2),

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\Delta_{10} a_{j,k+1}|}{k(k+1)} \leqslant \sum_{j=1}^{\infty} \sum_{q=2}^{\infty} |\Delta_{11} a_{jq}| \sum_{k=1}^{q-1} \frac{1}{k(k+1)} \leqslant \sum_{j=1}^{\infty} \sum_{q=2}^{\infty} |\Delta_{11} a_{jq}| < \infty.$$

Analogously, by (2.3),

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\Delta_{01} a_{j+1,k}|}{(j+1)k} &\leqslant \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+1)k} \sum_{p=j+1}^{\infty} |\Delta_{11} a_{pk}| \\ &\leqslant \sum_{p=2}^{\infty} \sum_{k=1}^{\infty} \frac{|\Delta_{11} a_{pk}|}{k} \sum_{j=1}^{p-1} \frac{1}{j+1} \\ &\leqslant \sum_{p=2}^{\infty} \sum_{k=1}^{\infty} \frac{|\Delta_{11} a_{pk}|}{k} \ln p < \infty \end{aligned}$$

and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{|a_{j+1,k+1}|}{(j+1)k(k+1)}$$

$$\leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(j+1)k(k+1)}\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}|\Delta_{11}a_{pq}|$$

$$=\sum_{p=2}^{\infty}\sum_{q=2}^{\infty}|\Delta_{11}a_{pq}|\sum_{j=1}^{p-1}\sum_{k=1}^{q-1}\frac{1}{(j+1)k(k+1)}$$

$$\leqslant \sum_{p=2}^{\infty}\sum_{q=2}^{\infty}|\Delta_{11}a_{pq}|\ln p<\infty.$$

Now, the proof of (3.1) is complete.

LEMMA 5. *If conditions* (1.1) *and* (2.3) *are satisfied, then the double series*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos jx \tag{3.2}$$

converges regularly for all $0<x\leqslant\pi$.

Proof. We have to prove that, for $0<x\leqslant\pi$,

$$S(p,m;q,n)=\sum_{j=p}^{m}\sum_{k=q}^{n}\frac{a_{jk}}{jk}\cos jx \qquad (p\leqslant m;\, q\leqslant n)$$

is arbitrarily small if $\max(p,q)$ is large enough. A summation by parts with respect to j yields

$$S(p,m;q,n)=\sum_{j=p}^{m-1}\sum_{k=q}^{n}\Delta_{10}\left(\frac{a_{jk}}{jk}\right)\left(\sum_{i=p}^{j}\cos ix\right)$$
$$+\sum_{k=q}^{n}\frac{a_{mk}}{mk}\left(\sum_{i=p}^{m}\cos ix\right).$$

Since

$$\Delta_{10}\left(\frac{a_{jk}}{jk}\right)=\frac{\Delta_{10}a_{jk}}{jk}+\frac{a_{j+1,k}}{j(j+1)k}$$

and, for $0<x\leqslant\pi$,

$$\left|\sum_{i=p}^{m}\cos ix\right|=\left|\frac{\sin(m+1/2)x-\sin(p-1/2)x}{2\sin(x/2)}\right|\leqslant\frac{\pi}{x}, \tag{3.3}$$

it is enough to see that, by (2.3),

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{|\Delta_{10}a_{jk}|}{jk}\leqslant\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{jk}\sum_{q=k}^{\infty}|\Delta_{11}a_{jq}|$$

$$\leqslant\sum_{j=1}^{\infty}\sum_{q=1}^{\infty}\frac{|\Delta_{11}a_{jq}|}{j}\sum_{k=1}^{q}\frac{1}{k}<\infty,$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{|a_{j+1,k}|}{j(j+1)k}\leqslant\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j(j+1)k}\sum_{p=j+1}^{\infty}\sum_{q=k}^{\infty}|\Delta_{11}a_{pq}|$$

$$=\sum_{p=2}^{\infty}\sum_{q=1}^{\infty}|\Delta_{11}a_{pq}|\sum_{j=1}^{p-1}\sum_{k=1}^{q}\frac{1}{j(j+1)k}$$

$$\leqslant 2\sum_{p=2}^{\infty}\sum_{q=1}^{\infty}|\Delta_{11}a_{pq}|\ln(q+1)<\infty,$$

and, finally,

$$\sup_{m\geqslant 1}\sum_{k=1}^{\infty}\frac{|a_{mk}|}{k}\leqslant\sum_{k=1}^{\infty}\frac{1}{k}\sum_{p=1}^{\infty}\sum_{q=k}^{\infty}|\Delta_{11}a_{pq}|$$

$$=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}|\Delta_{11}a_{pq}|\sum_{k=1}^{q}\frac{1}{k}<\infty.$$

4. Proof of Theorem 1

By (2.4), for every $\eta>0$ we can choose n so large that, for all $1\leqslant j_1\leqslant j_2$ and $1\leqslant k_1\leqslant k_2$,

$$\left|\sum_{j=j_1}^{j_2}\sum_{k=k_1}^{k_2}\frac{a_{jk}}{jk}\right|\leqslant\eta\qquad\text{if}\quad\max(j_1,k_1)\geqslant n.\tag{4.1}$$

By (1.1) and (1.2), series (1.4) converges uniformly on each rectangle $[\delta,\pi]\times[\varepsilon,\pi]$ where $0<\delta,\ \varepsilon<\pi$. Hence

$$\int_{\delta}^{\pi}\int_{\varepsilon}^{\pi}g(x,y)\,dx\,dy$$

$$=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}(\cos j\delta-\cos j\pi)(\cos k\varepsilon-\cos k\pi).\tag{4.2}$$

By Lemma 4, $\{a_{jk}/jk\}$ is a null sequence of bounded variation. Thus, the series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos jx\cos ky\qquad\text{converges}$$

for all $0<x,\ y\leqslant\pi$. Hence (2.5) is satisfied if we prove the existence of the limits

$$\lim_{\delta,\varepsilon\downarrow 0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos j\delta\cos k\varepsilon,$$

$$\lim_{\delta\downarrow 0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos j\delta\cos k\pi,$$

$$\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos j\pi\cos k\varepsilon.$$

First, we will prove that

$$\lim_{\delta,\varepsilon\downarrow 0}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos j\delta\cos k\varepsilon-\sum_{j=1}^{[1/\delta]}\sum_{k=1}^{[1/\varepsilon]}\frac{a_{jk}}{jk}\right)=0. \tag{4.3}$$

To this effect, let $\eta>0$ be given and let n be chosen according to (4.1). In the sequel, we assume that $0<\delta,\ \varepsilon<1/n$. Then

$$j_0=[1/\delta]\geqslant n \qquad\text{and}\qquad k_0=[1/\varepsilon]\geqslant n. \tag{4.4}$$

We decompose the double sums in (4.3) as follows

$$\begin{aligned}&\left(\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}+\sum_{j=n}^{j_0}\sum_{k=n}^{k_0}+\sum_{j=1}^{n-1}\sum_{k=n}^{k_0}+\sum_{j=n}^{j_0}\sum_{k=1}^{n-1}\right.\\&\quad\left.+\sum_{j=j_0+1}^{\infty}\sum_{k=k_0+1}^{\infty}+\sum_{j=1}^{j_0}\sum_{k=k_0+1}^{\infty}+\sum_{j=j_0+1}^{\infty}\sum_{k=1}^{k_0}\right)\frac{a_{jk}}{jk}\cos j\delta\cos k\varepsilon\\&=S_1+S_2+\cdots+S_7,\qquad\text{say;}\end{aligned}$$

and

$$\begin{aligned}&\left(\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}+\sum_{j=n}^{j_0}\sum_{k=n}^{k_0}+\sum_{j=1}^{n-1}\sum_{k=n}^{k_0}+\sum_{j=n}^{j_0}\sum_{k=1}^{n-1}\right)\frac{a_{jk}}{jk}\\&=T_1+T_2+T_3+T_4,\qquad\text{say.}\end{aligned}$$

Clearly,

$$T_1-S_1=\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}\frac{a_{jk}}{jk}(1-\cos j\delta\cos k\varepsilon)\to 0 \tag{4.5}$$

as $\delta,\ \varepsilon\downarrow 0$ provided n is fixed.

Applying Lemma 2 with $c_{jk} = 1 - \cos j\delta \cos k\varepsilon$ (observe that $\Delta_{11} c_{jk} \leqslant 0$ for $1 \leqslant j \leqslant j_0$ and $1 \leqslant k \leqslant k_0$) and using (4.1), we obtain that

$$|T_2 - S_2| \leqslant 2(1 - \cos^2 1) \max_{\substack{n \leqslant j_1 \leqslant j_0 \\ n \leqslant k_1 \leqslant k_0}} \left| \sum_{j=n}^{j_1} \sum_{k=n}^{k_1} \frac{a_{jk}}{jk} \right|$$

$$\leqslant 2(1 - \cos^2 1)\eta \tag{4.6}$$

which is small with η. In a similar way,

$$|T_r - S_r| \leqslant 2(1 - \cos 1)\eta \qquad \text{for} \quad r = 3, 4. \tag{4.7}$$

By Lemma 3, we can write $a_{jk} = b_{jk} - c_{jk}$ where b_{jk} and c_{jk} possess the properties indicated there. Then S_5 is the difference of two similar sums, one with b_{jk} and one with c_{jk}. By Lemma 2, the first of these sums

$$S_5' = \sum_{j=j_0+1}^{\infty} \sum_{k=k_0+1}^{\infty} \frac{b_{jk}}{jk} \cos j\delta \cos k\varepsilon$$

does not exceed in absolute value the quantity

$$\frac{b_{j_0+1, k_0+1}}{(j_0+1)(k_0+1)} \sup_{\substack{j_1 > j_0 \\ k_1 > k_0}} \left| \sum_{j=j_0+1}^{j_1} \sum_{k=k_0+1}^{k_1} \cos j\delta \cos k\varepsilon \right|.$$

Hence, by (3.3) and (4.4),

$$|S_5'| \leqslant \frac{b_{j_0+1, k_0+1}}{(j_0+1)(k_0+1)} \frac{1}{\sin(\delta/2) \sin(\varepsilon/2)} \leqslant \pi^2 b_{j_0+1, k_0+1}$$

which tends to zero as $\delta, \varepsilon \downarrow 0$ due to Lemma 3. The second part of S_5 is treated in the same way. Thus,

$$S_5 \to 0 \qquad \text{as} \quad \delta, \varepsilon \downarrow 0. \tag{4.8}$$

To estimate S_6 and S_7, we apply Lemma 3 again. As a result, S_6 and S_7 are also the difference of two similar sums, one with b_{jk} and one with c_{jk}, respectively. By Lemma 1,

$$|S_6'| = \left| \sum_{j=1}^{j_0} \cos j\delta \sum_{k=k_0+1}^{\infty} \frac{b_{jk}}{jk} \cos k\varepsilon \right|$$

$$\leqslant (\cos \delta) \max_{1 \leqslant i \leqslant j_0} \left| \sum_{j=1}^{i} \sum_{k=k_0+1}^{\infty} \frac{b_{jk}}{jk} \cos k\varepsilon \right|.$$

Since $\{\sum_{j=1}^{i} (b_{jk}/jk) : k=1, 2, ...\}$ is a nonincreasing null sequence, we can apply Lemma 1 again to get

$$\left|\sum_{k=k_0+1}^{\infty}\left(\sum_{j=1}^{i}\frac{b_{jk}}{jk}\right)\cos k\varepsilon\right|$$
$$\leqslant \sum_{j=1}^{i}\frac{b_{j,k_0+1}}{j(k_0+1)}\sup_{k_1>k_0}\left|\sum_{k=k_0+1}^{k_1}\cos k\varepsilon\right| \leqslant \pi\sum_{j=1}^{i}\frac{b_{j,k_0+1}}{j}.$$

Combining the last two inequalities and applying Lemma 3 give that

$$|S_6'| \leqslant \pi \sum_{j=1}^{j_0}\frac{b_{j,k_0+1}}{j} \leqslant \pi\sum_{j=1}^{j_0}\frac{1}{j}\sum_{p=j}^{\infty}\sum_{q=k_0+1}^{\infty}|\varDelta_{11}a_{pq}|$$
$$=\pi\sum_{p=1}^{\infty}\sum_{q=k_0+1}^{\infty}|\varDelta_{11}a_{pq}|\sum_{j=1}^{\min(j_0,p)}\frac{1}{j}$$

which tends to zero as $k_0\to\infty$ (or equivalently, $\varepsilon\downarrow 0$) due to (2.3). A similar estimate holds for the second part of S_6. So,

$$S_6\to 0 \qquad \text{as} \quad \delta, \varepsilon\downarrow 0. \tag{4.9}$$

An analogous reasoning applies to S_7 and results that

$$S_7\to 0 \qquad \text{as} \quad \delta, \varepsilon\downarrow 0. \tag{4.10}$$

Combining (4.5)–(4.10) yields (4.3) to be proved.

Second, we will prove that

$$\lim_{\delta\downarrow 0}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos j\delta\cos k\pi - \sum_{j=1}^{[1/\delta]}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos k\pi\right)=0. \tag{4.11}$$

Given $\eta>0$, we choose n in such a way that for all $1\leqslant j_1\leqslant j_2$

$$\left|\sum_{j=j_1}^{j_2}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}\cos k\pi\right|\leqslant\eta \qquad \text{if} \quad n\leqslant j_1\leqslant j_2. \tag{4.12}$$

This is possible by Lemma 5. Then we decompose the sums in (4.11) as follows

$$\left(\sum_{j=1}^{n-1}\sum_{k=1}^{\infty}+\sum_{j=n}^{j_0}\sum_{k=1}^{\infty}+\sum_{j=j_0+1}^{\infty}\sum_{k=1}^{\infty}\right)\frac{a_{jk}}{jk}\cos j\delta\cos k\pi$$
$$=U_1+U_2+U_3, \qquad \text{say;}$$

and

$$\left(\sum_{j=1}^{n-1} \sum_{k=1}^{\infty} + \sum_{j=n}^{j_0} \sum_{k=1}^{\infty}\right) \frac{a_{jk}}{jk} \cos k\pi = V_1 + V_2, \qquad \text{say.}$$

Clearly,

$$V_1 - U_1 = \sum_{j=1}^{n-1} (1 - \cos j\delta) \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \cos k\pi \to 0 \tag{4.13}$$

as $\delta \downarrow 0$ provided n is fixed. By Lemma 5, series (3.2) converges regularly for $x = \pi$, and consequently, each column $\sum_{k=1}^{\infty} (a_{jk}/jk) \cos k\pi$ also converges $(j = 1, 2, \ldots)$.

By Lemma 1 and (4.12),

$$\begin{aligned} |V_2 - U_2| &= \left| \sum_{j=n}^{j_0} (1 - \cos j\delta) \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \cos k\pi \right| \\ &\leqslant (1 - \cos 1) \max_{n \leqslant j_1 \leqslant j_0} \left| \sum_{j=n}^{j_1} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \cos k\pi \right| \\ &\leqslant (1 - \cos 1)\eta. \end{aligned} \tag{4.14}$$

Finally, we may assume that $\{a_{jk}\}$ is a nonincreasing null sequence (cf. Lemma 3). Then, by Lemma 2 and (4.4),

$$\begin{aligned} |U_3| &\leqslant \frac{a_{j_0+1,1}}{j_0+1} \sup_{j_1 > j_0} \left| \sum_{j=j_0+1}^{j_1} \sum_{k=1}^{\infty} \cos j\delta \cos k\pi \right| \\ &\leqslant \frac{a_{j_0+1,1}}{j_0+1} \frac{1}{\sin \delta/2} \leqslant \pi a_{j_0+1,1} \to 0 \end{aligned} \tag{4.15}$$

as $j_0 \to \infty$ (or equivalently, $\delta \downarrow 0$).

Putting (4.13)–(4.15) together gives (4.11) to be proved.

Third, we can prove along the same lines that

$$\lim_{\varepsilon \downarrow 0} \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \cos j\pi \cos k\varepsilon - \sum_{j=1}^{\infty} \sum_{k=1}^{[1/\varepsilon]} \frac{a_{jk}}{jk} \cos j\pi \right) = 0. \tag{4.16}$$

Collecting (4.2), (4.3), (4.11), and (4.16) completes the proof of Theorem 1.

5. Proof of Theorem 2

We introduce a new sequence $\{b_{mn} : m, n = 0, 1, ...\}$ by

$$b_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk}.$$

Then

$$b_{00} = \tfrac{1}{4} a_{00}, \qquad \Delta_{10} b_{m0} = -\tfrac{1}{2} a_{m+1,0},$$

$$\Delta_{01} b_{0n} = -\tfrac{1}{2} a_{0,n+1}, \qquad \Delta_{11} b_{mn} = a_{m+1,n+1} \quad (m, n = 0, 1, ...).$$

Thus, (2.10)–(2.12) imply that

$$b_{mn} \to 0 \qquad \text{as} \quad m + n \to \infty, \tag{5.1}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{11} b_{mn}| \ln(\max(m, n) + 2) < \infty. \tag{5.2}$$

We perform a double summation by parts in (1.3). As a result, we obtain that

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} (\cos mx - \cos(m+1)x)(\cos ny - \cos(n+1)y).$$

Even the weaker assumption that $\{b_{mn}\}$ is a null sequence of bounded variation is sufficient to do this. Hence

$$\frac{f(x, y)}{xy} = \frac{4 \sin(x/2) \sin(y/2)}{xy} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \sin\left(m + \frac{1}{2}\right) x \sin\left(n + \frac{1}{2}\right) y$$

and (2.14) is satisfied if and only if the sum of the series

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \sin\left(m + \frac{1}{2}\right) x \sin\left(n + \frac{1}{2}\right) y \\ &\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} \left(\sin mx \sin ny \cos\frac{x}{2} \cos\frac{y}{2} \right. \\ &\qquad + \sin mx \cos ny \cos\frac{x}{2} \sin\frac{y}{2} + \cos mx \sin ny \sin\frac{x}{2} \cos\frac{y}{2} \\ &\qquad \left. + \cos mx \cos ny \sin\frac{x}{2} \sin\frac{y}{2} \right) \\ &\quad = g_1(x, y) + g_2(x, y) + g_3(x, y) + g_4(x, y), \qquad \text{say,} \end{aligned} \tag{5.3}$$

is integrable on Q in the sense of improper Riemann integral.

First,

$$\lim_{\delta, \varepsilon \downarrow 0} \int_\delta^\pi \int_\varepsilon^\pi g_1(x, y)\, dx\, dy \qquad \text{exists} \tag{5.4}$$

if and only if

$$\lim_{\delta, \varepsilon \downarrow 0} \int_\delta^\pi \int_\varepsilon^\pi \sum_{m=1}^\infty \sum_{n=1}^\infty b_{mn} \sin mx \sin ny\, dx\, dy \qquad \text{exists.}$$

By (5.1) and (5.2), Theorem 1 applies and (5.4) follows from (2.10).

Second,

$$\lim_{\delta, \varepsilon \downarrow 0} \int_\delta^\pi \int_\varepsilon^\pi g_2(x, y)\, dx\, dy \qquad \text{exists} \tag{5.5}$$

if and only if

$$\lim_{\delta, \varepsilon \downarrow 0} \int_\delta^\pi \int_\varepsilon^\pi \sum_{m=1}^\infty \sum_{n=0}^\infty b_{mn} \sin mx \cos ny \sin \frac{y}{2}\, dx\, dy \qquad \text{exists.} \tag{5.6}$$

By single summation by parts, we get that

$$\begin{aligned}
&\sum_{m=1}^\infty \sum_{n=0}^\infty b_{mn} \sin mx \cos ny \sin \frac{y}{2} \\
&\quad = \sum_{n=0}^\infty \left(\sum_{m=1}^\infty (\Delta_{01} b_{mn}) \sin mx \right) \sum_{k=0}^n \cos ky \sin \frac{y}{2} \\
&\quad = \frac{1}{2} \sum_{m=1}^\infty \sum_{n=0}^\infty (\Delta_{01} b_{mn}) \sin mx \left(\sin \frac{y}{2} + \sin \left(n + \frac{1}{2} \right) y \right).
\end{aligned}$$

Hence the integral in (5.6) equals

$$\frac{1}{2} \sum_{m=1}^\infty \sum_{n=0}^\infty (\Delta_{01} b_{mn}) \frac{\cos m\delta - \cos m\pi}{m} \left(2 \cos \frac{\varepsilon}{2} + \frac{\cos(n+1/2)\varepsilon}{n+1/2} \right).$$

We claim that the limit of this series exists as $\delta, \varepsilon \downarrow 0$. To this effect, we show its absolute convergence, uniformly in δ and ε. In fact, by (5.1) and (5.2),

$$\begin{aligned}
\sum_{m=1}^\infty \sum_{n=0}^\infty \frac{|\Delta_{01} b_{mn}|}{m} &\leqslant \sum_{m=1}^\infty \sum_{n=0}^\infty \frac{1}{m} \sum_{p=m}^\infty |\Delta_{11} b_{pn}| \\
&= \sum_{p=1}^\infty \sum_{n=0}^\infty |\Delta_{11} b_{pn}| \sum_{m=1}^p \frac{1}{m} < \infty.
\end{aligned}$$

This proves ultimately (5.5).

Third, the case of $g_3(x, y)$ can be treated in a similar way.

Fourth, by a double summation by parts,

$$g_4(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\Delta_{11} b_{mn}) \sum_{j=0}^{m} \cos jx \sin \frac{x}{2}$$
$$\cdot \sum_{k=0}^{n} \cos ky \sin \frac{y}{2}$$
$$= \frac{1}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\Delta_{11} b_{mn}) \left(\sin \left(m + \frac{1}{2} \right) x + \sin \frac{x}{2} \right)$$
$$\cdot \left(\sin \left(n + \frac{1}{2} \right) y + \sin \frac{y}{2} \right).$$

It follows from this and (5.2) that the integral

$$\int_0^{\pi} \int_0^{\pi} g_4(x, y)\, dx\, dy \qquad \text{exists} \tag{5.7}$$

in the sense of (absolute) Lebesgue integral.

Combining (5.4), (5.5), and its symmetric counterpart for $g_3(x, y)$, (5.7) yields (2.14) to be proved.

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